QUANTIFICATION AND INTERACTION

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Contents

- Standard quantification (history, linguistic data)
- Models, generalized quantifiers
- Second order and individual concepts
- What is a quantifier (in proof theory)?
  - Generic elements (Hilbert)
  - Cut-elimination
- Conclusion

USUAL QUANTIFICATION

Some, a, there is,…

All, each, any, every,…

Aristotle,
& scholastics (Avicenna, Scott, Ockham)

- *A* and *B* are terms
  (« term » is vague: middle-age distinction bewteen terms, « suppositionnes », eg. Ockham)
  1. All *A* are *B*
  2. Some *A* are *B*
  3. No *A* are *B*
  4. Not all *A* are *B*
  - Rules, syllogisms
  - Remarks:
    - Little about models or truth condition
    - Always a restriction, sorts, kinds,
    - « not all » is not lexicalized and some *A* are not *B* has a different focus.

FREGE AND ANALYTIC PHILOSOPHY

- After the algebraic computational approach of Leibniz, Boole, De Morgan, Pierce,…
- Predicate calculus, first order logic
  for instance distinction between
  $\forall x\ (A(x) \rightarrow (B(x) \vee C(x))$
  $\forall x\ (A(x) \rightarrow B(x)) \vee \forall x\ (A(x) \rightarrow C(x))$
- Attempt of a deductive system
- A single universe where variables vary:
  - All A are B
  - $\forall x(A(x) \rightarrow B(x))$

THE ADEQUATION BETWEEN PROOFS AND MODELS

- Deduction, proofs (Hilbert)
  using a generic element
- Models, truth condition (Tarski)
- Adequation proofs-models:
  completeness theorem (Gödel, Herbrand, ~1930)
  - Whatever is provable is true in any model.
  - What is true in every model is provable.
- This results holds
  - For classical logic
    Extensions are possible (intuitionistic, modal,…)
  - For first order logic
    No satisfying extension.
  - For usual quantification
    No proper deductive system for generalized quantifiers

HOW DOES ONE ASSERT , USE OR REFUTE USUAL QUANTIFIED SENTENCES

- In classical logic, *reductio ad absurdum, tertium no datur,* can be used.
- Otherwise:
  - « Exists » introduction rule
    - (how to prove ∃ as a conclusion) :
    - if for some object *a* *P(a)* is proved, then we may infer ∃*x* *P(x)*
  - « Exists » elimination rule
    - (how to use ∃ as an assumption):
    - if we know that ∃*xP(x)*, and that C holds under the assumption *P(a)* with an *a* which is never present elsewhere, we may infer C without the assumption P(a).

HOW DOES ONE ASSERT , USE OR REFUTE USUAL QUANTIFIED SENTENCES

- « For all » introduction rule
  - (how to prove ∀ as a conclusion)
  - To establish ∀*xP(x)*, one has to show *P(a)* for an object *a* without any particular property, i.e. a generic object *a*.
  - If the domain is known, one can conclude ∀*xP(x)* from a proof of *P(a)* for each object *a* of the domain. The domain has to be finite to keep proofs finite. The Omega rule of Gentzen is an exception.
- « For all » elimination rule
  - (how to use ∀ as an assumption)
  - From ∀*xP(x)*, one can conclude *P(a)* for any object *a*.

Refutations

- How do we refute usual quantification?
- $\exists x P(x)$: little can be done apart from proving that all do not have the property.
- $\forall x P(x)$: ***Any dog may bite.***
  this can be refuted in at least two ways:
  - Displaying an object not satisfying P
    Rex would never bite.
  - Asserting that a subset does not satisfy P, thus remainig with generic elements:
    Basset hounds do not bite.
- This is related to the Avicennian idea that a property of a term (individual or not) is always asserted for the term as part of a class:
  it is more related to type theory than to the Fregean view of a single universe.

USUAL QUANTIFICATION IN ORDINARY LANGUAGE EXISTENTIALS

- Existential are highly common
  Discourse is often structured according to existentials as in Discourse Representation Theory.
- They can be with or without restriction, but in the later case the restiction is implicit: human beings, things, …
  - There's a tramp sittin' on my doorstep
  - Some girls give me money
  - Something happened to me yesterday
- Focus:
  - Some politicians are crooks. (youtube)
  - ? Some crooks are politicians.

USUAL QUANTIFICATION IN ORDINARY LANGUAGE UNIVERSALS

- Less common but present.
- With or without restriction:
  - Everyone, everything, anyone, anything,…
  - Every, all, each,…
- Generic (proofs), distributive (models)
  - Whoever, every,
  - All, each,
- Sometimes ranges over potentially infinite sets:
  - Each star in the sky is an enormous glowing ball of gas.
  - All groups of stars are held together by gravitational forces.

USUAL QUANTIFICATION IN ORDINARY LANGUAGE UNIVERSAL NEGATIVE

- With or without restriction:
  - No one, nothing, not any, …
  - No,…
- Generic or distributive:
  - Because no planet's orbit is perfectly circular, the distance of each varies over the course of its year.
  - Nothing's gonna change my world.
  - Porterfield went where no colleague had gone previously this season, realising three figures.

Usual quantification in ordinary language existential negative

- Not lexicalised (in every human language?):
  - Not all, not every + NEG
  - Alternative formulation (different focus): some … are not … / some … do not …
- Harder to grasp (psycholinguistic tests), frequent misunderstandings
- Rather generic reading:
  - Not Every Picture Tells a Story
  - Everyone is *entitled* to an opinion, but *not every* opinion is *entitled* to student government funding.
- Alternative formulation (different focus):
  - *Some Students Do Not* Participate In Group Experiments Or Projects.

Individual concepts

Alternative view of individuals and quantification

MOTIVATION
FOR INDIVIDUAL CONCEPTS

- Usual semantics with possible worlds:
  It is impossible to believe that
  Tullius≠Cicero
  with rigid designator
- To comme back to the notion of TERM
  - Individuals are particular cases of predicates.
- Quantification is a property of predicates.

First order in second order: proofs

- P is an individual concept whenever IC(P):
  - $\forall x \forall y (P(x) \wedge P(y) \rightarrow x=y)$
  - Exists x P(x)
- First order quantification
  from second order quantification:
  - Forall P IC(P) implies X(P)
  - Exists P IC(P) and X(P)
- As far as proofs are concerned, this is equivalent to first order quantification – and when non emptyness is skipped one only as implication with first order quantification. (Lacroix & Ciardelli)

MODELS?

- Natural (aka principal models): no completeness
- Henkin models:
  completeness and compactness
  but unnatural,
  e.g. one satisfies all the following formulae:
  - F_0: every injective map is a bijection (Dedekind finite)
  - F_n, $n \geq 1$: there are at least n elements

Generalized quantifiers

Quite common in natural language

Central topic in analytic philosophy (models)

Proofs and refutations?

DEFINITION

- Generalized quantifiers are operators that gives a proposition from two properties (two unary predicates):
  - A restriction
  - A predicate
- Some are definable from usual first order logic:
  - At most two,
  - Exactly three
- And some are not (from compactness):
  - The majority of…
  - Few /a few …
  - Most of… (strong majority + vague)
- Observe that Frege's reduction cannot apply:
  - Most students go out on Thursday evening.
  - For most people, if they are student then they go out on Thursday evening

Models / proofs

- There are many studies about the models, the properties of such quantifiers, in particular monotony w.r.t. the restriction or the predicate.
- Some assertion about cardinality are wrong:
  - Most numbers are not prime. Can be found in maths textbooks.
  - Test on "average" people:
    - most number are prime (no)
    - most number are not prime (yes)
  - No cardinality but measure, and what would be the corresponding generic element? An object enjoying most of the properties?
- Little is known about the proofs (tableaux methods without specific rules, but taking the intended model into account).

« MOST OF », « THE MAJORITY OF » REMARKS

- *Most of* is distinct from *the majority of*:
  - The majority of French people voted
    - for Chirac in 2002 (82%).
    - for Sarkozy in 2007 (53%).
  - Most of French people voted
    - for Chirac in 2002. (82%)
    - * for Sarkozy in 2007. (53%)
- The percentage for « most of » to hold is contextual.
- Most of is a vague quantifier.

« The majority of » Attempt (proof vs. Refutation)

- Two ways of refuting
  the majority of (meaning at least 50%) the A have the property P:
  - Only the minority of the A has the property P
  - There is another property Q
    which hold for the majority of the A
    with no A satisfying P and Q.
  - What would be a generic majority element?

DEFINE JOINTLY RULES FOR:
1) THE MAJORITY OF
2) A MINORITY OF

- « For all » entails the « majority of »
- If any property Q which is true of the majority of Ameets P, then P holds for the majority of the A (impredicative definition, needs further study)
- A minority of A is NOT P
  should be equivalent to
  The majority of A is P
- The majority of does not entail a minority of
- Forall => majority of
- Only a minority => Exists
- *A linguistic remark why do we say « The majority » but « A minority »*

WHAT IS A QUANTIFIER?

Proof-theretical analysis:

Tools to allow the communication (cut) between proofs

COMMUNICATION (INTERACTION) BETWEEN PROOFS: CUT RULE

- Cut-rule: two proofs π and ρ may communicate (interact) by means of a formula *A*, i.e. when
  - π ends with a formula *A* and other formulas Γ
  - ρ ends with the negation ~*A* of *A* and other formulas Λ
- The communication (interaction) between such a pair of proofs produces a proof which ends with the formulas Γ and the formulas Λ
- Cut-elimination procedure: is the development of such a communication (interaction)
- Interaction: ~*A* is the negation of *A*, and *A* is the negation of ~*A* .

PARTICULAR CASE (INTUITIONISTIC COMMUNICATION)

- Cut: communication between a proof π of the conclusion A from the assumptions Γ (i.e. a proof which ends with *A* and the negation of the formulas Γ) and a proof ρ of a conclusion *C* from the assumption *A* and the assumptions Λ (i.e. a proof which ends with *C*, the negation of *A* and the negation of Λ)
- The communication between such a pair of proofs produces a proof of the conclusion *C* from the assumptions Γ and the assumptions Λ (i.e. a proof which ends with *C*, the negation of the formulas Γ and the negation of the formulas Λ).

A SPECIAL CASE OF COMMUNICATION, LEADING TO QUANTIFIERS.

- A proof π which ends with a formula *A(b)* and formulas Γ
- A proof ρ which ends with a formula *~A(d)* and formulas Λ
- These proofs may communicate (cut) when one of these cases hold:
  - The object b is the same as the object *d* (indeed, replace *b* by *d* in *A(b)*, or replace *d* by *b* in *~A(d)*)
  - The object b is generic in π (i.e. it does not occur in the formulas Γ) (indeed, replace *b* by *d* in *A(b)*
  - The object d is generic in ρ (i.e. it does not occur in the formulas Λ) (indeed, replace *d* by *b* in *~A(d)*)

GENERIC OBJECTS : HILBERT'S APPROACH, 1

- Name of generic objects (no quantifier)
  Rules for these names
- Express the fact that b is a generic object in the formula $A(b)$ (in a proof π), in one of these two ways
  - b is an object such that, if b has the property A then every object has the property A

    $$\tau x\ A(x)$$

  - b is an object such that, if some object has not the property A, then b has not the property A

    $$\varepsilon x\ {\sim}A(x)$$

GENERIC OBJECTS : HILBERT'S APPROACH, 2

- Rules for τx:
  - From *A(b)* with *b* generic in a proof π, infer *A(τxA(x))*
  - From *~A(d)* , infer *~A(τxA(x))*
  - So, one reduces to general case of cut rule
  - The development of cut rule is: replace *τxA(x)* by *d*
- Rules for εx:
  - From *A(b)* with b generic in a proof π, infer *A(εx~A(x))*
  - From *~A(d)* , infer *~ A(εx~A(x))*
  - So, one reduces to general case of cut rule
  - The development of cut rule is: replace *εx~A(x)* by *d*
- …

GENERIC OBJECTS: FREGE'S APPROACH

- Forget generic objects by means of operators ∀,∃
  Rules of operators ∀,∃
- New formulas: $\forall x A(x)$, $\exists x A(x)$, with

$$\sim\forall x A(x) = \exists x \sim A(x)$$

- Rules of operators ∀,∃ :
  - Rule of ∀ :
    from $A(b)$ with b generic object, infer $\forall x A(x)$
  - Rule of ∃ : from $\sim A(d)$, infer $\exists x \sim A(x)$
  - So, reduces to the general case of cut rule
  - The development of cut rule will be replace the generic object b by d.

THE APPROACHES ARE EQUIVALENT. ONLY 2 QUANTIFIERS?

- The following equivalences hold:
  - $\forall x A(x) \leftrightarrow A(\tau x A(x))$
  - $\forall x A(x) \leftrightarrow A(\varepsilon x {\sim} A(x))$
  - "Universal quantification"
- The following equivalence hold:
  - $\exists x A(x) \leftrightarrow A(\varepsilon x A(x))$
  - $\exists x A(x) \leftrightarrow A(\tau x {\sim} A(x))$
  - "Existential quantification"

THE TWO DEFINITIONS ARE NOT EQUIVALENT FOR GENERALIZED QUANTIFIERS

- Observe that the Fregean definition of quantifiers with a single universe is not possible with generalized quantifiers:
  - Most student go out on Thursday nights.
  - For most people if they are students then they go out on Thursday nights.
- But still we can ask whether it is possible to introduce other quantifiers, in this proof-theoretical way.

NEW QUANTIFIERS, FROM A PROOF-THEORETICAL POINT OF VIEW

- A way inspired by Non Commutative Linear Logic where new (multiplicative and non commutative) connectives are added to the usual ones
- Introduce a pair of quantifiers,
  a variant $\forall^*$ of $\forall$, and a variant $\exists^*$ of $\exists$.
- Decide one of the following two possibilities:
  - $\forall^* x A(x)$ implies $\forall x A(x)$ and so $\exists x A(x)$ implies $\exists^* x A(x)$
  - $\exists^* x A(x)$ implies $\exists x A(x)$ and so $\forall x A(x)$ implies $\forall^* x A(x)$
  - (the second one is more natural…)
- In both the cases, one of new quantifiers is obtained by adding a new rule, the other one is obtained by restricting the rule.
- …
- May we define in this way the quantifier
  "the majority of x" or "most *x* have the property *A*" …

Conclusion

Of this preliminary work

Rules for (Generalized) quantifiers

- Which properties of quantifier rules guarantee that they behave properly in proofs and interaction?
- Is it possible to define a proof system for some generalized quantifiers?
  - Percentage?
  - Vague quantifiers?
  - …
- What are the corresponding notions of generic elements?

PREDICATION,SORTS AND QUANTIFICATION

- How do we take into account the sorts,what linguist call the restriction of the quantifier (in a typed system, a kind of ontology)?
- To avoid a paradox of the Fregean single sort:
  - Garance is not tall (as a person, for opening the fridge).
  - Garance is tall (for a two year old girl).
- One quantifier per type or a general quantifier which specializes?
  On type theory it would be a constant of the system F: ForAll/Exists: $\Pi X ((X \rightarrow t) \rightarrow t)$

Thanks

Any question?