



QUANTIFICATION AND INTERACTION

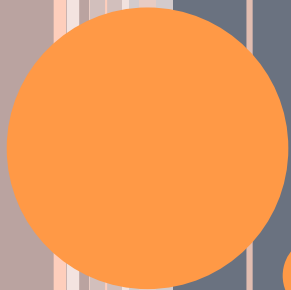
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CONTENTS

- Standard quantification (history, linguistic data)
- Models, generalized quantifiers
- Second order and individual concepts
- What is a quantifier (in proof theory)?
 - Generic elements (Hilbert)
 - Cut-elimination
- Conclusion



USUAL QUANTIFICATION



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Some, a, there is,...

All, each, any, every,...



ARISTOTLE, & SCHOLASTICS (AVICENNA, SCOTT, OCKHAM)

- *A* and *B* are terms
(« term » is vague: middle-age distinction between terms, « suppositionnes », eg. Ockham)
 1. All *A* are *B*
 2. Some *A* are *B*
 3. No *A* are *B*
 4. Not all *A* are *B*
- Rules, syllogisms
- Remarks:
 - Little about models or truth condition
 - Always a restriction, sorts, kinds,
 - « not all » is not lexicalized and some *A* are not *B* has a different focus.

FREGE AND ANALYTIC PHILOSOPHY

- After the algebraic computational approach of Leibniz, Boole, De Morgan, Pierce,...
- Predicate calculus, first order logic
for instance distinction between
$$\forall x (A(x) \rightarrow (B(x) \vee C(x)))$$
$$\forall x (A(x) \rightarrow B(x)) \vee \forall x (A(x) \rightarrow C(x))$$
- Attempt of a deductive system
- A single universe where variables vary:
 - All A are B
 - $\forall x(A(x) \rightarrow B(x))$

THE ADEQUATION BETWEEN PROOFS AND MODELS

- Deduction, proofs (Hilbert)
using a generic element
- Models, truth condition (Tarski)
- Adequation proofs-models:
completeness theorem (Gödel, Herbrand, ~1930)
 - Whatever is provable is true in any model.
 - What is true in every model is provable.
- This results holds
 - For classical logic
Extensions are possible (intuitionistic, modal,...)
 - For first order logic
No satisfying extension.
 - For usual quantification
No proper deductive system for generalized quantifiers

HOW DOES ONE ASSERT , USE OR REFUTE USUAL QUANTIFIED SENTENCES

- In classical logic, *reductio ad absurdum*, *tertium no datur*, can be used.
- Otherwise:
 - « Exists » introduction rule
 - (how to prove \exists as a conclusion):
 - if for some object a $P(a)$ is proved, then we may infer $\exists x P(x)$
 - « Exists » elimination rule
 - (how to use \exists as an assumption):
 - if we know that $\exists x P(x)$, and that C holds under the assumption $P(a)$ with an a which is never present elsewhere, we may infer C without the assumption $P(a)$.

HOW DOES ONE ASSERT , USE OR REFUTE USUAL QUANTIFIED SENTENCES

- « For all » introduction rule
 - (how to prove \forall as a conclusion)
 - To establish $\forall xP(x)$, one has to show $P(a)$ for an object a without any particular property, i.e. a generic object a .
 - If the domain is known, one can conclude $\forall xP(x)$ from a proof of $P(a)$ for each object a of the domain. The domain has to be finite to keep proofs finite. The Omega rule of Gentzen is an exception.
- « For all » elimination rule
 - (how to use \forall as an assumption)
 - From $\forall xP(x)$, one can conclude $P(a)$ for any object a .

REFUTATIONS

- How do we refute usual quantification?
- $\exists xP(x)$: little can be done apart from proving that all do not have the property.
- $\forall xP(x)$: ***Any dog may bite.***
this can be refuted in at least two ways:
 - Displaying an object not satisfying P
Rex would never bite.
 - Asserting that a subset does not satisfy P,
thus remaining with generic elements:
Basset hounds do not bite.
- This is related to the Avicennian idea that a property of a term (individual or not) is always asserted for the term as part of a class:
it is more related to type theory than to the Fregean view of a single universe.

USUAL QUANTIFICATION IN ORDINARY LANGUAGE EXISTENTIALS

- Existentials are highly common
Discourse is often structured according to existentials as in Discourse Representation Theory.
- They can be with or without restriction, but in the latter case the restriction is implicit: human beings, things, ...
 - There's a tramp sittin' on my doorstep
 - Some girls give me money
 - Something happened to me yesterday
- Focus:
 - Some politicians are crooks. (youtube)
 - ? Some crooks are politicians.

USUAL QUANTIFICATION IN ORDINARY LANGUAGE UNIVERSALS

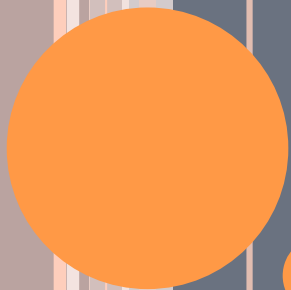
- Less common but present.
- With or without restriction:
 - Everyone, everything, anyone, anything,...
 - Every, all, each,...
- Generic (proofs), distributive (models)
 - Whoever, every,
 - All, each,
- Sometimes ranges over potentially infinite sets:
 - Each star in the sky is an enormous glowing ball of gas.
 - All groups of stars are held together by gravitational forces.

USUAL QUANTIFICATION IN ORDINARY LANGUAGE UNIVERSAL NEGATIVE

- With or without restriction:
 - No one, nothing, not any, ...
 - No,...
- Generic or distributive:
 - Because no planet's orbit is perfectly circular, the distance of each varies over the course of its year.
 - Nothing's gonna change my world.
 - Porterfield went where no colleague had gone previously this season, realising three figures.

USUAL QUANTIFICATION IN ORDINARY LANGUAGE EXISTENTIAL NEGATIVE

- Not lexicalised (in every human language?):
 - Not all, not every + NEG
 - Alternative formulation (different focus):
some ... are not ... / some ... do not ...
- Harder to grasp (psycholinguistic tests),
frequent misunderstandings
- Rather generic reading:
 - Not Every Picture Tells a Story
 - Everyone is *entitled* to an opinion, but *not every* opinion is *entitled* to student government funding.
- Alternative formulation (different focus):
 - *Some Students Do Not Participate In Group Experiments Or Projects.*



INDIVIDUAL CONCEPTS

Alternative view of individuals and quantification

MOTIVATION FOR INDIVIDUAL CONCEPTS

- Usual semantics with possible worlds:
It is impossible to believe that
 Tullius≠Cicero
with rigid designator
- To come back to the notion of TERM
 - Individuals are particular cases of predicates.
- Quantification is a property of predicates.

FIRST ORDER IN SECOND ORDER: PROOFS

- P is an individual concept whenever $IC(P)$:
 - $\forall x \forall y (P(x) \wedge P(y) \rightarrow x=y)$
 - $\exists x P(x)$
- First order quantification from second order quantification:
 - For all P $IC(P)$ implies $X(P)$
 - $\exists P IC(P)$ and $X(P)$
- As far as proofs are concerned, this is equivalent to first order quantification – and when non emptyness is skipped one only as implication with first order quantification. (Lacroix & Ciardelli)

MODELS?

- Natural (aka principal models): no completeness
- Henkin models:
completeness and compactness
but unnatural,
e.g. one satisfies all the following formulae:
 - F_0 : every injective map is a bijection
(Dedekind finite)
 - $F_n, n \geq 1$: there are at least n elements



GENERALIZED QUANTIFIERS



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Quite common in natural language

Central topic in analytic philosophy (models)



Proofs and refutations?

DEFINITION

- Generalized quantifiers are operators that gives a proposition from two properties (two unary predicates):
 - A restriction
 - A predicate
- Some are definable from usual first order logic:
 - At most two,
 - Exactly three
- And some are not (from compactness):
 - The majority of...
 - Few /a few ...
 - Most of... (strong majority + vague)
- Observe that Frege's reduction cannot apply:
 - Most students go out on Thursday evening.
 - For most people, if they are student then they go out on Thursday evening

MODELS / PROOFS

- There are many studies about the models, the properties of such quantifiers, in particular monotony w.r.t. the restriction or the predicate.
- Some assertion about cardinality are wrong:
 - Most numbers are not prime.
Can be found in maths textbooks.
 - Test on “average” people:
 - most number are prime (no)
 - most number are not prime (yes)
 - No cardinality but measure, and what would be the corresponding generic element? An object enjoying most of the properties?
- Little is known about the proofs (tableaux methods without specific rules, but taking the intended model into account).

« MOST OF », « THE MAJORITY OF » REMARKS

- *Most of* is distinct from *the majority of*:
 - The majority of French people voted
 - for Chirac in 2002 (82%).
 - for Sarkozy in 2007 (53%).
 - Most of French people voted
 - for Chirac in 2002. (82%)
 - * for Sarkozy in 2007. (53%)
- The percentage for « most of » to hold is contextual.
- Most of is a vague quantifier.

« THE MAJORITY OF » ATTEMPT (PROOF VS. REFUTATION)

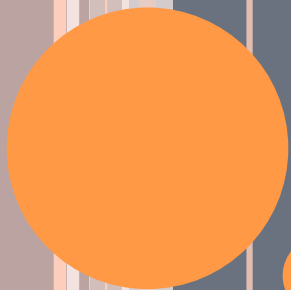
- Two ways of refuting the majority of (meaning at least 50%) the A have the property P:
 - Only the minority of the A has the property P
 - There is another property Q which hold for the majority of the A with no A satisfying P and Q.
 - What would be a generic majority element?

DEFINE JOINTLY RULES FOR:

1) THE MAJORITY OF

2) A MINORITY OF

- « For all » entails the « majority of »
- If any property Q which is true of the majority of A meets P, then P holds for the majority of the A (impredicative definition, needs further study)
- A minority of A is NOT P should be equivalent to The majority of A is P
- The majority of does not entail a minority of
- For all \Rightarrow majority of
- Only a minority \Rightarrow Exists
- *A linguistic remark why do we say « The majority » but « A minority »*



WHAT IS A QUANTIFIER?



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Proof-theoretical analysis:

Tools to allow the communication (cut) between proofs



COMMUNICATION (INTERACTION) BETWEEN PROOFS: CUT RULE

- Cut-rule: two proofs π and ρ may communicate (interact) by means of a formula A , i.e. when
 - π ends with a formula A and other formulas Γ
 - ρ ends with the negation $\sim A$ of A and other formulas Λ
- The communication (interaction) between such a pair of proofs produces a proof which ends with the formulas Γ and the formulas Λ
- Cut-elimination procedure: is the development of such a communication (interaction)
- Interaction: $\sim A$ is the negation of A , and A is the negation of $\sim A$.

PARTICULAR CASE (INTUITIONISTIC COMMUNICATION)

- Cut: communication between a proof π of the conclusion A from the assumptions Γ (i.e. a proof which ends with A and the negation of the formulas Γ) and a proof ρ of a conclusion C from the assumption A and the assumptions Λ (i.e. a proof which ends with C , the negation of A and the negation of Λ)
- The communication between such a pair of proofs produces a proof of the conclusion C from the assumptions Γ and the assumptions Λ (i.e. a proof which ends with C , the negation of the formulas Γ and the negation of the formulas Λ).

A SPECIAL CASE OF COMMUNICATION, LEADING TO QUANTIFIERS.

- A proof π which ends with a formula $A(b)$ and formulas Γ
- A proof ρ which ends with a formula $\sim A(d)$ and formulas Λ
- These proofs may communicate (cut) when one of these cases hold:
 - The object b is the same as the object d (indeed, replace b by d in $A(b)$, or replace d by b in $\sim A(d)$)
 - The object b is generic in π (i.e. it does not occur in the formulas Γ) (indeed, replace b by d in $A(b)$)
 - The object d is generic in ρ (i.e. it does not occur in the formulas Λ) (indeed, replace d by b in $\sim A(d)$)

GENERIC OBJECTS : HILBERT'S APPROACH, 1

- Name of generic objects (no quantifier)
Rules for these names
- Express the fact that b is a generic object in the formula $A(b)$ (in a proof π), in one of these two ways

- b is an object such that, if b has the property A then every object has the property A

$$\tau x A(x)$$

- b is an object such that, if some object has not the property A , then b has not the property A

$$\varepsilon x \sim A(x)$$

GENERIC OBJECTS : HILBERT'S APPROACH, 2

- Rules for τx :
 - From $A(b)$ with b generic in a proof π , infer $A(\tau x A(x))$
 - From $\sim A(d)$, infer $\sim A(\tau x A(x))$
 - So, one reduces to general case of cut rule
 - The development of cut rule is: replace $\tau x A(x)$ by d
- Rules for εx :
 - From $A(b)$ with b generic in a proof π , infer $A(\varepsilon x \sim A(x))$
 - From $\sim A(d)$, infer $\sim A(\varepsilon x \sim A(x))$
 - So, one reduces to general case of cut rule
 - The development of cut rule is: replace $\varepsilon x \sim A(x)$ by d
- ...

GENERIC OBJECTS: FREGE'S APPROACH

- Forget generic objects by means of operators \forall, \exists
Rules of operators \forall, \exists
- New formulas: $\forall xA(x)$, $\exists xA(x)$, with
$$\sim\forall xA(x) = \exists x\sim A(x)$$
- Rules of operators \forall, \exists :
 - Rule of \forall :
from $A(b)$ with b generic object, infer $\forall xA(x)$
 - Rule of \exists : from $\sim A(d)$, infer $\exists x\sim A(x)$
 - So, reduces to the general case of cut rule
 - The development of cut rule will be replace the generic object b by d .

THE APPROACHES ARE EQUIVALENT. ONLY 2 QUANTIFIERS?

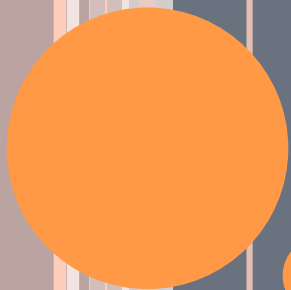
- The following equivalences hold:
 - $\forall x A(x) \leftrightarrow A(\tau x A(x))$
 - $\forall x A(x) \leftrightarrow A(\varepsilon x \sim A(x))$
 - “Universal quantification”
- The following equivalence hold:
 - $\exists x A(x) \leftrightarrow A(\varepsilon x A(x))$
 - $\exists x A(x) \leftrightarrow A(\tau x \sim A(x))$
 - “Existential quantification”

THE TWO DEFINITIONS ARE NOT EQUIVALENT FOR GENERALIZED QUANTIFIERS

- Observe that the Fregean definition of quantifiers with a single universe is not possible with generalized quantifiers:
 - Most student go out on Thursday nights.
 - For most people if they are students then they go out on Thursday nights.
- But still we can ask whether it is possible to introduce other quantifiers, in this proof-theoretical way.

NEW QUANTIFIERS, FROM A PROOF-THEORETICAL POINT OF VIEW

- A way inspired by Non Commutative Linear Logic where new (multiplicative and non commutative) connectives are added to the usual ones
- Introduce a pair of quantifiers, a variant \forall^* of \forall , and a variant \exists^* of \exists .
- Decide one of the following two possibilities:
 - $\forall^*xA(x)$ implies $\forall xA(x)$ and so $\exists xA(x)$ implies $\exists^*xA(x)$
 - $\exists^*xA(x)$ implies $\exists xA(x)$ and so $\forall xA(x)$ implies $\forall^*xA(x)$
 - (the second one is more natural...)
- In both the cases, one of new quantifiers is obtained by adding a new rule, the other one is obtained by restricting the rule.
- ...
- May we define in this way the quantifier “the majority of x ” or “most x have the property A ” ...



CONCLUSION

Of this preliminary work

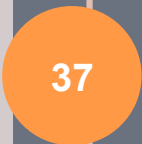
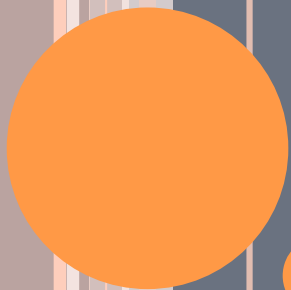
RULES FOR (GENERALIZED) QUANTIFIERS

- Which properties of quantifier rules guarantee that they behave properly in proofs and interaction?
- Is it possible to define a proof system for some generalized quantifiers?
 - Percentage?
 - Vague quantifiers?
 - ...
- What are the corresponding notions of generic elements?

PREDICATION, SORTS AND QUANTIFICATION

- How do we take into account the sorts, what linguist call the restriction of the quantifier (in a typed system, a kind of ontology)?
- To avoid a paradox of the Fregean single sort:
 - Garance is not tall (as a person, for opening the fridge).
 - Garance is tall (for a two year old girl).
- One quantifier per type or a general quantifier which specializes?

On type theory it would be a constant of the system F : ForAll/Exists: $\Pi X ((X \rightarrow t) \rightarrow t)$



THANKS

Any question?