Records in Ludics

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Context: Ludics and Dialogue theories

- One of the main aims of LOCI is the development of a theory of dialogue and its applications for linguistic phenomenons in the ludics framework.
- Until now we mainly considered one aspect of Ludics : "it is a theory of interaction". But, in order to deepen our formalisation we also have to exploit its relevance to represent computationnal objects.

This talk:

- to introduce some properties of Ludics, the ones which seem relevant to deal with computationnal theories
- to describe the proposition to deal with the notion of “record” in the ludical framework as it was already set in the seminal article of Girard [Locus Solum, 2001]
- to expose the transposition of such objects : record, in *computationnal Ludics*. A framework due to K. Terui in which the objects and concepts of Ludics are set in a relevant way to a computationnal treatment.

Ludics: a theory of logic

Ludics is a theory elaborated by J-Y. Girard to reconstruct logic starting from the notion of *interaction*.

- The logical main notions: *formulas*, *proofs* are not " a priori" given but recovered, "rebuilt" from the notion of **interaction** (the cut-rule).
- Ludics was developed starting from concepts of Linear Logic:
  - **polarisation**: linear logic rules have polarities, either $+$ or $-$, thus making proofs sequences of polarized steps.
  - **focalisation**: results coming from Theoretical Computer Science [**?**] lead us to focalized proofs, that is proofs as alternating sequences of steps.

Such an alternance is close to the one of plays in a game.
Ludics may be seen from a game semantics approach.

In Ludics the notion of proof is subsumed by the one of *design* which may be seen as a proof search or as a strategy.

As a strategy a design is a set of plays (**chronicles**)
The plays themselves are alternating sequences of moves (**actions**).

Actions as moves

An action (move) is given by three datas:
- a **polarity** (one player's view being fixed, this player's moves are positive, his/her opponent's are negative),
- a **focus**, that is the location (*locus*) of the move, and
- a **ramification**, which represents the finite set of locations which can be reached in one step.

A special positive move is provided by the so called **daïmon**, denoted by †. (which will enable to define the winning position of the opponent)

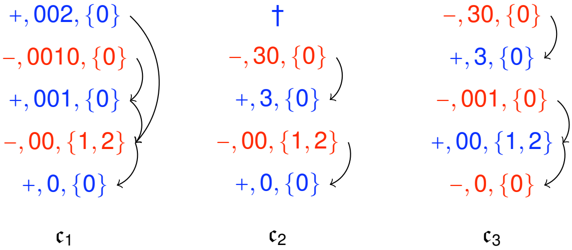
Chronicle: an alternated sequence of actions

$+, 002, \{0\}$	$\dagger$	$-, 30, \{0\}$
$-, 0010, \{0\}$	$-, 30, \{0\}$	$+, 3, \{0\}$
$+, 001, \{0\}$	$+, 3, \{0\}$	$-, 001, \{0\}$
$-, 00, \{1, 2\}$	$-, 00, \{1, 2\}$	$+, 00, \{1, 2\}$
$+, 0, \{0\}$	$+, 0, \{0\}$	$-, 0, \{0\}$
$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{c}_3$

In a chronicle the positive actions are:
- either justified $((+, 001, \{0\})$; $(+, 002, \{0\}))$
- or initial $((+, 0, \{0\})$ or $(+, 3, \{0\}))$.

the negative actions are justified by the immediate previous action (except the starting one).

The base of a chronicle:



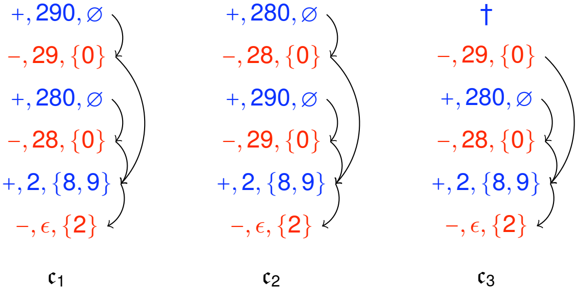
The **base** of the chronicle is a sequent $\Gamma \vdash \Delta$ of loci where the set Γ contains all the initial focus of a negative action (so at most one) and Δ contains all the initial focus of a positive action (so a finite number).

$\mathfrak{c}_1$ is based on $\vdash 0$ (which is a positive base);

$\mathfrak{c}_2$ is based on $\vdash 0, 3$ (which is also a positive base);

$\mathfrak{c}_3$ is based on $0 \vdash 3$ (which is a negative base).

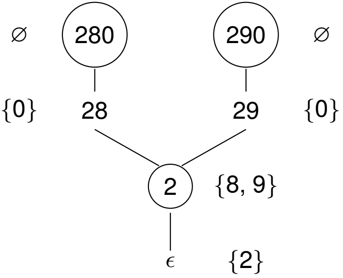
Coherence relation on chronicles



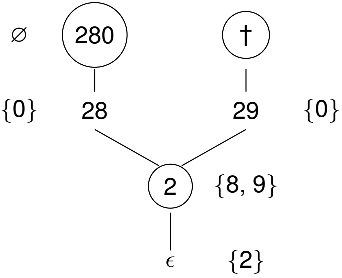
Two chronicles $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are **coherent**, when one may put them together in the same strategy.

In the foregoing example, the chronicles are pairwise coherent except that $\mathfrak{c}_1$ and $\mathfrak{c}_3$.

Examples of strategies based on $\epsilon \vdash$



The strategy $\mathfrak{D}_1$
contains the chronicles $\mathfrak{c}_1$ and $\mathfrak{c}_2$



The strategy $\mathfrak{K}_1$
contains the chronicles $\mathfrak{c}_3$ and $\mathfrak{c}_2$

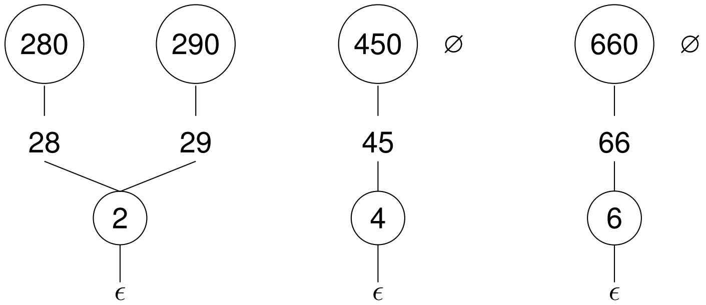
They are both based on $\epsilon \vdash$.

(Designs)

A **design** $\mathfrak{D}$*, based on* $\Gamma \vdash \Delta$*, is a set of chronicles based on* $\Gamma \vdash \Delta$*, such that the following conditions are satisfied:*

- **The chronicles are pairwise coherent.**
- *The set is prefix closed.*
- *A chronicle without extension in* $\mathfrak{D}$ *ends with a positive action.*
- $\mathfrak{D}$ *is non-empty when the base is positive (in that case all the chronicles begin with a unique positive action).*

Examples: some designs based on $\epsilon \vdash$



$\mathfrak{D}_1 = \{\mathfrak{c}_1, \mathfrak{c}_2, \dots\}$

$\mathfrak{D}_2 = \{\mathfrak{d}, \dots\}$

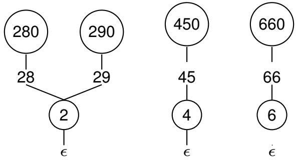
$\mathfrak{D}_3 = \{\mathfrak{e}, \dots\}$

Strategy-like versus proof-like presentation

The design $\mathfrak{D} = \mathfrak{D}_1 \cup \mathfrak{D}_2 \cup \mathfrak{D}_3$ is also a design based on $\epsilon \vdash$.

Strategy-like design

$\mathfrak{D} =$



Proof-like design

$$\mathfrak{D} = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 280}\emptyset}{28 \vdash} \quad \dfrac{\dfrac{}{\vdash 290}\emptyset}{29 \vdash}}{\vdash 2} \quad \dfrac{\dfrac{\dfrac{}{\vdash 450}\emptyset}{45 \vdash}}{\vdash 4} \quad \dfrac{\dfrac{\dfrac{}{\vdash 660}\emptyset}{66 \vdash}}{\vdash 6}}{\epsilon \vdash}$$

Proof-like presentation : design as dessin

A **design** *(as dessin) based on* $\Gamma \vdash \Delta$ *is a tree of sequents built by means of the three following rules:*

- DAÏMON

$$\frac{}{\vdash \Delta}\dagger$$

- POSITIVE RULE

$$\frac{\ldots \xi.i \vdash \Delta_i \ldots}{\vdash \Delta, \xi}(\xi, I)$$

- NEGATIVE RULE

$$\frac{\cdots \vdash \xi.I, \Delta_I \ldots}{\xi \vdash \Delta}(\xi, \mathfrak{N})$$

(Nets, cut-net, closed cut-net)

- *A* **net** *is a finite set of designs on disjoint bases.*
- *A* **cut-net** *is a net with* **cuts***: an addresse present only once in a negative position of a base and once in a positive one of another base.*

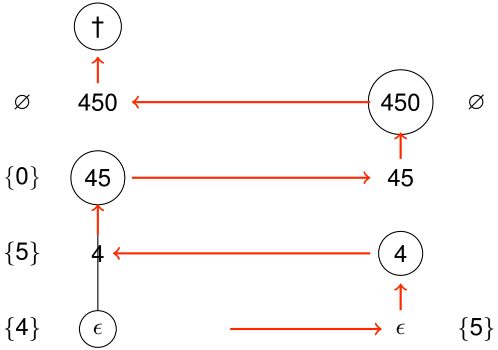
  Example : $\dfrac{\mathfrak{D}_1}{\xi \vdash \sigma} \quad \dfrac{\mathfrak{D}_2}{\sigma \vdash \rho, \tau} \quad \dfrac{\mathfrak{D}_3}{\rho \vdash \alpha}$
- *In a* **closed cut-net** *all addresses in bases are parts of some cut.*

  Example : $\dfrac{\mathfrak{F}_0}{\vdash \xi} \quad \dfrac{\mathfrak{F}_1}{\xi \vdash}$

Interaction in closed case

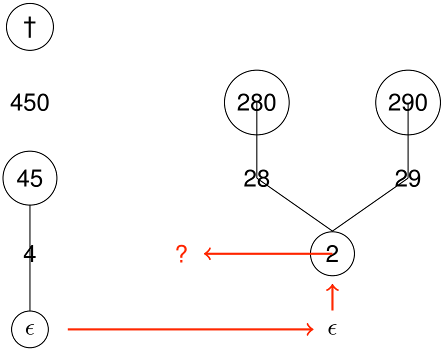
Informel definition: It simply corresponds to a travel which starts from the first positive node
and, applies, at each proper positive action, the following:
moves to the corresponding negative one, (if there is one, if not, interaction fails),
moves upward to the unique action which follows, and so on,
and if the travel meets †, the interaction successfully terminates on it.

The interaction may terminate like the normalization between $\mathfrak{D}_2$ and $\mathfrak{F}$:



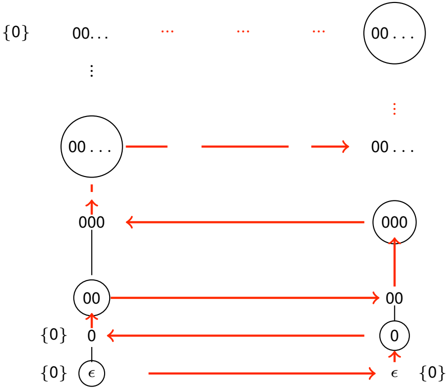
It simply corresponds to a travel which starts from the first positive node, moves to the corresponding negative one, moves upward to the unique action which follows, and so on, and successfully terminates on †.

The interaction may fail like the normalization between $\mathfrak{D}_1$ and $\mathfrak{F}$:



the travel starts from the first positive node at each proper positive action, moves to the corresponding negative one, if there is one. Otherwise the normalization fails !

The interaction may not terminate



At each step an action $(+, 0\dots, \{0\})$ meets the action $(-, 0\dots, \{0\})$ and the interaction continues. . .

(Orthogonality)

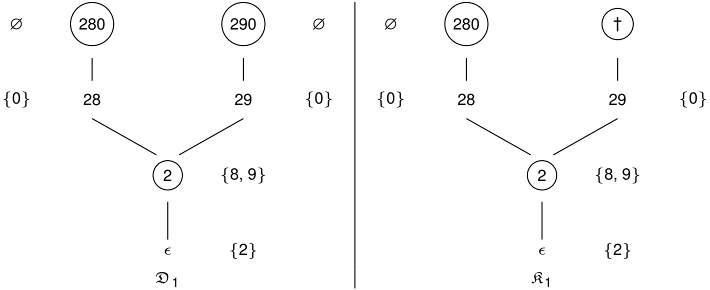
Two designs $\mathfrak{D}$ and $\mathfrak{E}$ respectively based on $\vdash \xi$ and $\xi \vdash$, are said othogonal when $[[\mathfrak{D}, \mathfrak{E}]] = \{\dagger\}$.
$\mathfrak{D}^{\perp}$ denotes the set of all designs $\mathfrak{E}$ such that $\mathfrak{D}$ and $\mathfrak{E}$ are orthogonal.

From the notion of orthogonality: an order relation

It is then possible to compare two designs according to their counter-designs:

$$\mathfrak{D} \leq \mathfrak{S} \text{ iff } \mathfrak{D}^{\perp} \subset \mathfrak{S}^{\perp}$$

Example:



$$\mathfrak{D}_1 \leq \mathfrak{K}_1$$

Another reason for being more defined

$$\mathfrak{D}_1 = \dfrac{\dfrac{\dfrac{\overline{\vdash 280}^{\,\emptyset}}{28 \vdash} \qquad \dfrac{\overline{\vdash 290}^{\,\emptyset}}{29 \vdash}}{\vdash 2}}{\epsilon \vdash}$$

$$\mathfrak{D} = \dfrac{\dfrac{\dfrac{\overline{\vdash 280}^{\,\emptyset}}{28 \vdash} \qquad \dfrac{\overline{\vdash 290}^{\,\emptyset}}{29 \vdash}}{\vdash 2} \qquad \dfrac{\dfrac{\overline{\vdash 450}^{\,\emptyset}}{45 \vdash}}{\vdash 4} \qquad \dfrac{\dfrac{\overline{\vdash 660}^{\,\emptyset}}{66 \vdash}}{\vdash 6}}{\epsilon \vdash}$$

$$\mathfrak{D}_1 \preceq \mathfrak{D}$$

Formulas/Types

(Behaviour)

Let E be a set of designs on same base, E is a behaviour *if* $E = E^{\perp\perp}$.
A behaviour is positive *(resp.* negative*) if the base of its designs is positive (resp. negative).*

Some connectives

Let M and N be two negative behaviours on the same base $\xi \vdash$.
Let P and Q be some positive behaviour on the same base $\vdash \xi$.

- $M \& N := M \cap N$.
- $P \oplus Q := (P \cup Q)^{\perp\perp}$.
  $P \otimes Q = \{(+, \xi, I \cup J)w$ such that:
  $(+, \xi; I)w \in \mathfrak{D} \in P \ \ (+, \xi, J)w \in \mathfrak{E} \in Q$ and $I \cap J = \varnothing\}^{\perp\perp}$.
- The neutral element of the multiplicative conjonction 1, on the base $\vdash \xi$, is the behaviour:
  $\{\{(+, \xi, \varnothing)\}\}^{\perp\perp}$.

Example

Design

$$\dfrac{\dfrac{\dfrac{}{\vdash 280}\emptyset}{28 \vdash} \qquad \dfrac{\dfrac{}{\vdash 290}\emptyset}{29 \vdash}}{\dfrac{\vdash 2}{\epsilon \vdash}}$$

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash} \qquad \dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash}}{\vdash \uparrow 1 \otimes \uparrow 1}}{\downarrow (\uparrow 1 \otimes \uparrow 1)^{\perp} \vdash}$$

The design $\mathfrak{D}_1$ belongs to the negative behaviour $M_1 = \uparrow (\uparrow 1 \otimes \uparrow 1)$.

Another types

Let $M = \uparrow 1 \otimes \uparrow 1 \;\&\; \downarrow\uparrow 1 \;\&\; \downarrow\uparrow 1$.

Design

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 280}\emptyset}{28 \vdash} \quad \dfrac{\dfrac{}{\vdash 290}\emptyset}{29 \vdash}}{\vdash 2} \quad \dfrac{\dfrac{\dfrac{}{\vdash 450}\emptyset}{45 \vdash}}{\vdash 4} \quad \dfrac{\dfrac{\dfrac{}{\vdash 660}\emptyset}{66 \vdash}}{\vdash 6}}{\epsilon \vdash}$$

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash} \quad \dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash}}{\vdash \uparrow 1 \otimes \uparrow 1} \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash}}{\vdash \downarrow\uparrow 1} \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash}}{\vdash \downarrow\uparrow 1}}{M^{\perp} \vdash}$$

The design $\mathfrak{D}$ belongs to the behaviour M.
And it belongs also to the behaviour M_1.

Incarnation and connectives

(Incarnation)

Let B be a behaviour, $\mathfrak{D}$ be a design in B.

- $|\mathfrak{D}|_B = \bigcap\{\mathfrak{R} \subset \mathfrak{D}$ *and* $\mathfrak{R} \in B\}$ *is the* incarnation *of* $\mathfrak{D}$ *with respect to the behaviour B.*
- *A design* $\mathfrak{D}$ *is* material *in a behaviour B if* $\mathfrak{D} = |\mathfrak{D}]_B$.
- *The* incarnation *of a behaviour B, denoted $|B|$, is the set of its material designs.*

Let M and N be two negative behaviours on the same base:

$$M \& N = M \cap N \quad |M \& N| = |M| \times |N|$$

Let P and Q be two positive behaviours on the same base:

$$P \oplus Q = (P \cup Q)^{\perp\perp} \quad P \oplus Q = P \cup Q \cup \{\mathfrak{D}ai\}$$

Examples

Design

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 280}\emptyset}{28 \vdash} \quad \dfrac{\dfrac{}{\vdash 290}\emptyset}{29 \vdash}}{\vdash 2} \quad \dfrac{\dfrac{\dfrac{}{\vdash 450}\emptyset}{45 \vdash}}{\vdash 4} \quad \dfrac{\dfrac{\dfrac{}{\vdash 660}\emptyset}{66 \vdash}}{\vdash 6}}{\epsilon \vdash}$$

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash} \quad \dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash}}{\vdash \uparrow 1 \otimes \uparrow 1} \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash}}{\vdash \downarrow\uparrow 1} \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash}}{\vdash \downarrow\uparrow 1}}{M^{\perp} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 280}\emptyset}{28 \vdash} \quad \dfrac{\dfrac{}{\vdash 290}\emptyset}{29 \vdash}}{\vdash 2}}{\epsilon \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash} \quad \dfrac{\dfrac{}{\vdash 1}\emptyset}{\downarrow 1^{\perp} \vdash}}{\vdash \uparrow 1 \otimes \uparrow 1}}{M_1^{\perp} \vdash}$$

The design $\mathfrak{D}$ belongs to the behaviour M and to the behaviour M_1.
The design $\mathfrak{D}$ is material w.r.t M but is not material w.r.t M_1.
The design $\mathfrak{D}_1$ is material w.r.t. the behaviour M_1.

Records: an example

Suppose you want represent the record:

$$\left(\begin{array}{c} coord = (8,9) \\ colour = green \\ shape = cercle \end{array} \right)$$

The atomic date types are :

- the type : $Colour = \oplus_{i=1}^{8} \downarrow\uparrow 1_i$ based on $\vdash 4$
  for the colours, where the data "green" is encoded by $i = 5$;
- the type : $Shape = \oplus_{i=1}^{3} \downarrow\uparrow 1_i$ based on $\vdash 6$
  for the shapes, where the data "cercle" is encoded by $i = 6$;
- the type : $Coord = \oplus_{i=1}^{\infty} \uparrow 1_i \otimes \oplus_{k=1}^{\infty} \uparrow 1_k$ based on $\vdash 2$
  for the coordinates, where the data "(n,m)" is encoded by $i = n$ and $k = m$.

We then use the type $R = \uparrow Coord \,\&\, \uparrow Shape \,\&\, \uparrow Colour$, to type the datas :

$$\begin{pmatrix} coord = (8,9) \\ colour = green \\ shape = cercle \end{pmatrix}$$

And the record $(coord = (8,9); colour = green; shape = cercle)$ is the design $\mathfrak{D}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 280}\emptyset}{28 \vdash} \quad \dfrac{\dfrac{}{\vdash 290}\emptyset}{29 \vdash}}{\vdash 2} \quad \dfrac{\dfrac{\dfrac{}{\vdash 450}\emptyset}{45 \vdash}}{\vdash 4} \quad \dfrac{\dfrac{\dfrac{}{\vdash 660}\emptyset}{66 \vdash}}{\vdash 6}}{\epsilon \vdash}$$

which is material in R.

We also have that:
$|\mathfrak{D}|_{\downarrow Coord} = \mathfrak{D}_1$; $|\mathfrak{D}|_{\downarrow Colour} = \mathfrak{D}_2$; $|\mathfrak{D}|_{\downarrow Shape} = \mathfrak{D}_3$.
Where,

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{}{\vdash 280}\emptyset}{28 \vdash} \quad \dfrac{\dfrac{}{\vdash 290}\emptyset}{29 \vdash}}{\vdash 2} \\ \epsilon \vdash \\ \mathfrak{D}_1
\end{array}
\qquad
\begin{array}{c}
\dfrac{\dfrac{\dfrac{}{\vdash 450}\emptyset}{45 \vdash}}{\vdash 4} \\ \hline \epsilon \vdash \\ \mathfrak{D}_2
\end{array}
\qquad
\begin{array}{c}
\dfrac{\dfrac{\dfrac{}{\vdash 660}\emptyset}{66 \vdash}}{\vdash 6} \\ \hline \epsilon \vdash \\ \mathfrak{D}_3
\end{array}
$$

If we are interested only in one field, for example the colour of the object, we just need to consider its incarnation w.r.t. $\downarrow Colour$.

C-Ludics

Terui's c-designs extend Girard's designs as $\lambda-$terms generalised.

- At the place of
$$(\lambda x M)N \to M[N/x]$$
we write
$$\lambda(x).M|\overline{\lambda} < N > \to M[N/x]$$
- We replace the couple $\lambda/\overline{\lambda}$ of λ-calculus with the couple $a/\overline{a}$ for any name $a \in A$

$\mathcal{A}$ is a **signature** i.e. the pair (A, ar) where A is a set of names and $ar : A \to \mathbb{N}$ is a function which gives an arity to each name of A.

C-designs

The c-designs are co-inductively defined as

- *Negative c-designs are defined as* $x \mid \sum_{a \in A} a(\vec{x}_a).P_a$
- *Positive c-designs are defined as* $✠ \mid \Omega \mid N_o|\overline{a} < N_1, \ldots, N_n >$

where for each name $a \in A$ P_a is a positive c-design, and for $i = 0, \ldots, n \quad N_i$ is a negative c-design.

- $\sum_{a \in A} a(\vec{x}_a).P_a$ corresponds to $\lambda^a \vec{x}_a.Pa$
- $N_o|\overline{a} < N_1, \ldots, N_n >$ corresponds to $(\lambda^a \vec{x}_a.P_a)N_1 \ldots N_n$

- A **cut** is a c-design of the form $\sum_{a \in A} a(\vec{x}).P_a | \overline{a} < N_1, \ldots, N_n >$.
- A variable x occurring as $N_o | \overline{a} < N_1, \ldots, x, \ldots, N_n >$ in a c-design T is called an **identity** in T.
  If $T = x$ T is itself an identity.

The reduction relation

To eliminate cuts in c-designs we define a reduction relation.

*The **reduction relation** $\rightarrow$ is defined on positive c-designs as follows*

$$(\sum a(\vec{x}_a).P_a)\overline{a} < \vec{N} > \rightarrow P_a[\vec{N}/\vec{x}_a]$$

- *We write $P \Downarrow Q$ if P reduces to Q in a finite numer of steps. (i.e. $P \rightarrow^{\star} Q$) and Q is neither a cut nor Ω (P converges to Q).*
- *Otherwise we write $P \Uparrow$ (P diverges).*

Orthogonality

- A c-design is **closed** if it has no free variables.
- A positive c-design P is **atomic** if it has got at most one free variable (called x_0). A negative c-design N is atomic if it doesn't have free variables.
- An atomic, positive c-design P and an atomic,negative one N are **orthogonal** and written $P \perp N$ if substituting N to x_0 in P we obtain a c-design which converges to ✠ i.e. $P[N/x_0] \Downarrow$✠.

Order relations

The **observational ordering** $\preccurlyeq$ is the largest binary relation R over c-designs such that

1. if $\maltese\ R\ T$ then $T = \maltese$
2. if $\Omega\ R\ T$ then T is positive
3. if $N_o|\overline{a} < N_1, \ldots, N_n > R\ T$ then $T = \maltese$ or $T = M_o|\overline{a} < M_1, \ldots, M_n >$ and $N_i\ R\ M_i$ for $i = 0, \ldots, n$
4. If $x\ R\ T$ then $T = x$.
5. if $(\sum a(\vec{x_a}).P_a)\ R\ T$ then $T = \sum a(\vec{x_a}).Q_a$ with $P_a\ R\ Q_a$ for every $a \in A$

Observational ordering corresponds to the preorder relation over designs i.e.
Given two designs $\mathfrak{D}, \mathfrak{E}$ that correspond to the c-designs d, e if $\mathfrak{D} \preccurlyeq \mathfrak{E}$ then $d \preccurlyeq e$.

Now let's talk about a class of c-designs that corresponds to Girard's original designs.

Standard c-designs

A c-design T is ***standard*** *if it is*

- *total ($T \neq \Omega$)*
- *The set of free variables of T, $fv(T)$ is finite*
- *linear i.e. for any subdesign of T of the form $N_0|a < N_1, \dots N_n >$ the sets $fv(N_0), \dots fv(N_n)$ are pairwise disjoint.*
- *cut-free*
- *identity-free*

Standard c-designs correspond to original designs of Girard over the signature **RAM** $= (Pf(\mathbb{N}), |\quad|)$ where $|\quad|$ gives the cardinality of $I \in Pf(\mathbb{N})$) and viceversa.

Strong separation

A standard c-design T admits ***strong separation*** *if there is an anti-design $[T^c]$ such that $T \perp [T^c]$ and $U \not\perp [T^c]$ for any standard, $\maltese$-free c-design U such that $T \npreceq U$.*

This means that there is an anti-design which characterizes T by orthogonality .

data-designs

In order to represent data (as for instance natural numbers) Terui introduces a set of negative c-designs called data-designs.
We consider a signature $\mathcal{A}$ that contains a fixed unary name $\uparrow$.

The set of data designs consists of negative c-designs, (with the only negative action $\uparrow$*), coinductively defined as* $d = \uparrow (x).x|\overline{a} < d_1, \ldots, d_n >$ *where* $x \notin fv(d_i)$ *for* $i = 1, \ldots, n$.

Fixed a 0-ary name *zero* and an unary name *suc* we have
$0^\star = \uparrow (x).x|zero$
$(n+1)^\star = \uparrow (x).x|suc < n^\star >$

Properties of data-designs

Data designs satisfy some properties.

- data-designs are stardard
- every finite data design can be duplicated
- any pair of distinct data designs are incomparable with respect to $\preccurlyeq$ hence $d_1 \preccurlyeq d_2$ implies $d_1 = d_2$ for any data designs d_1, d_2.
- all finite data-designs enjoy strong separation
- functions over data-designs are defined as follows.

*An **n-any function design** is a negative c-design $F[x_1, \ldots, x_n]$ such that $fv(F) \subseteq \{x_1, \ldots, x_n\}$ and $[[F[d_1, \ldots, d_n]]]$ is either a data-design or $\uparrow (x).\Omega$ for any data design $d_1, \ldots, d_n$*

The result of **strong separation over finite data designs** can be expressed as follows

Let d be a finite data-design. For any negative standard c-design N which is ✠-free $(d)^c_{x_0} \perp N \Leftrightarrow d \preccurlyeq N$.

where given a finite data-design d, the positive atomic c-desing $(d)^c_{x_0}$ is defined adding a ✠ when the visit of d is terminated.

Representation of records

We want to represent records using data designs, but this definition of data designs isn't enough because the only negative action involved is $\uparrow(x)$, and records can contain more then one field. We need more negative actions.

- So we fix a signature A which contains a name a for every field of a record that we want to represent.
- We define the set of data-record designs that is an extension of the set of data-designs, adding a negative action $a(x)$ for every field of a record that we want to represent.

Data-record designs

*The set of **data-record designs** consists of negative designs $d, d_1, \ldots, d_n$ is coinductively defined as follows*

$$d = \sum_{c_1,\ldots,c_k} c_i(x_i).x_i|\overline{a} < d_1, \ldots, d_n >$$

The record **coord : (3,4)**, **colour : green**, **shape : circle** is represented by the following data-record design

$$coord(x).x|\overline{(3,4)}^2 <\uparrow (x).x|\overline{3}^0, \uparrow (x).x|\overline{4}^0 > +colour(y).y$$

$$|\overline{green}^1 <\uparrow (x).x|\overline{green}^0 > +shape(t).t|\overline{circle}^1 <\uparrow (x).x|\overline{circle}^0 >$$

What happened to properties?

Differently from data designs is possible to compare two data-record designs i.e. $d \preccurlyeq e \not\Rightarrow d = e$ for any d, e data-record designs.
For instance let e be a data-record designs which represents the record *colour* : *blue*, *shape* : *circle*, and d the data-record design which represents the record *colour* : *blue* i.e.

$d = colour(x).x|\overline{blue^1} <\uparrow (y).y|\overline{blue^0} > +shape(t).\Omega$ and
$e = colour(x).x|\overline{blue^1} <\uparrow (y).y|\overline{blue^0} > +shape(t).t|\overline{circle^1} <\uparrow (z).z|\overline{circle^0} >$
$d \preccurlyeq e$ and $d \neq e$.

Functions over data-record designs

We extend the definition of function over data-designs to data-record designs as follows.

*An **n-any function-record** is a negative c-design $F[x_1, \ldots, x_n]$ such that $fv(F) \subseteq \{x_1, \ldots, x_n\}$ and $[[F[d_1, \ldots, d_n]]]$ is either a data-record design or $\sum a(\vec{x}_a).\Omega$ for any data-record design $d_1, \ldots, d_n$*

Example 1 : $F_{shape}[x]$

Let's define a function $F_{shape}[x]$ which taken a data-record design d of the type *coord* & *colour* & *shape* isolates the part of d which represents the field shape.
$d = coord(x).x|\overline{(m,n)^2} <\uparrow (z).z|\overline{m^0}, \uparrow (z).z|\overline{n^0} >$
$+colour(y).y|\overline{c^1} <\uparrow (z).z|\overline{c^0} > +shape(t).t|\overline{s^1} <\uparrow (z).z|\overline{s^0} >$

We set

$$F_{shape}[x] = shape(t).(x|\overline{shape} < \sum a(x_1,\ldots,x_n).t|\overline{a} < x_1,\ldots,x_n >>)$$

$F_{shape}[x]$ is not identity-free satisfies the definition of n-ary function-record.

Example 2 : $G[x]$

$G[x]$ is such that taken a data-record design d as above, controls the part of d which represent the field colour and if the colour is blue, it isolates the part of d which represents the field shape; otherwise it diverges i.e.

If c^1 represents the colour blue then
$G[d] \Downarrow shape(t).t|\overline{s^1} <\uparrow (z).z|\overline{s^0}>$ otherwise
$G[d] \Downarrow shape(t).\Omega$.

$$G[x] = shape(t).(x|\overline{colour} < blue(y).(x|\overline{shape} < \mathfrak{Fax}_t >)$$

$G[x]$ is not linear, then is not standard. But this doesn't create any problem with the definition of n-ary function-record. Indeed $fv(G[x]) \subseteq \{x\}$ and for any data-record design d $[[G[d]]]$ is a data-record design or the divergence ($shape(t).\Omega$).

Strong separation for data-record designs?

Differently from data-record ones, finite data-record designs don't always admit strong separation, i.e. there is a finite data-record design d such that there is no anti-design $[d^c]$ such that $d \perp [d^c]$ and $N \pm [d^c]$ for any standard, ✠-free c-design N such that $d \npreceq N$.
For instance let d, N', N'' be data-record designs which represents respectively the records
colour : *blue*, *shape* : *circle*
colour : *blue*
shape : *circle*
then $d \npreceq N'$ and $d \npreceq N''$ but there is no anti-design d' such that $d' \perp d$, $d' \pm N'$ and $d' \pm N''$.

strong separation and incarnation

We go back to data-designs, consider the behavior $d^\perp$ and its incarnation $|d^\perp|$, and observe that $(d)^c_{x_0} \in (d)^\perp$ and for every negative, standard, $\maltese$-free c-design d' if $d \preccurlyeq d'$ $((d)^\perp \subseteq (d')^\perp)$ then $d' \perp (d)^c_{x_0}$. So we wonder if $d^\perp$ can be generated by the c-design $(d)^c_{x_0}$ i.e. $|d^\perp| = \{(d)^c_{x_0}\}^{\maltese}$. This would mean : if a c-design $T \in d^\perp \Rightarrow (d)^c_{x_0} \preccurlyeq T$.

This is not true.
Indeed if we consider the data-design which represents the record *coord* : $(8, 9)$
$d = \uparrow (x).x|\overline{8, 9} < \uparrow (y).y|\overline{8^0}, \uparrow (y).y|\overline{9^0} >$
According to the definition
$(d)^c_{x_0} = x_0| \downarrow < \{8, 9\}(x_1, x_2).x_1| \downarrow < \{8^0\}.x_2| \downarrow < \{9^0\}.\maltese >>>$.

The c-design
$e = x_0 | \downarrow< \{8,9\}(x_1, x_2).x_2 | \downarrow< \{9^0\}.x_1 | \downarrow< \{8^0\}.\maltese>>>$ is orthogonal to d but $(d)^c_{x_0} \not\preccurlyeq e$.
$(d)^c_{x_0}$ and e correspond to two designs which only differs for the order of some rules.

It looks possible for $d^{\perp}$ to be generated by a certain (finite) number of c-designs (the number depends on the structure of $d_1, \ldots, d_n$ if $d = \uparrow (x).x | a < d_1, \ldots, d_n >$).
We are looking for a weaker version of strong separability for data-record designs.
Maybe the following?

- Given a finite data-record design d there are m positive c-designs $(d)^1_{x_0}, \ldots, (d)^m_{x_0}$ such that $(d)^i_{x_0} \perp d$ for $i = 1, \ldots, m$ and for every standard $\maltese$-free c-design d'' such that $d \nleq d''$ there is $i \in \{1, \ldots, m\}$ such that $(d)^i_{x_o} \not\perp d''$